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Closed form solution for acoustic wave equation between two parallel plates using Euler–Maclaurin sum formula

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Abstract

The Euler–Maclaurin sum formula is applied to the infinite series of images in the Green function for acoustic wave propagation between two perfectly reflecting parallel plates. The major part of the series is represented with an integral representation by the formula. The formula also provides a reminder. An approximate closed form solution is obtained for the Green function, including both an exact major part and an approximate remainder. It is applied to the transient response for an exponentially decaying point acoustic source located between the plates. For a specific numerical example, the approximate closed form solution is compared to infinite series summation.

1. Introduction

The acoustic wave equation for a point source $v(t)$ is the partial differential equation for pressure $p(x, y, x, t)$ (see Ref. [1]):

$$\nabla^2_{xyz} p(x, y, x, t) - \frac{1}{c^2}\frac{\partial^2 p(x, y, z, t)}{\partial t^2} = \delta_{xyz} v(t), \tag{1}$$

where ∇^2_{xyz} is the three-dimensional Laplacian operator and c is the speed of sound. The solution to the acoustic wave equation between two parallel plates is well documented for the case of a source modelled as a band of noise. For a band of noise, the infinite series of images that are generated by the plates are considered as incoherent sources allowing the summation to be carried out on an energy basis without consideration of phase. For example, Bies and Hansen [2] used the method of images to give the following closed form result for mean square pressure a distance r

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from a point source between two perfectly reflecting plates with spacing H.

$$\langle p^2\rangle_t \propto \sum_{n=-\infty}^{\infty} \frac{1}{r^2+(nH)^2} \to \frac{\pi}{Hr}\coth\left(\frac{\pi r}{H}\right), \quad \text{for large } \frac{r}{H}. \tag{2}$$

The closed form result is valid only for large r/H values. For other values a computer calculation is appropriate since the series is convergent for all r. The band of noise source case is prevalent in the literature because it represents the condition in many areas of acoustics such as industrial factory spaces and open plan offices. Transient acoustics requires a closer look at the wave equation because the images due to the reflecting plates cannot be considered incoherent. For transient response, the direct computer calculation of the summation generally requires many terms for convergence. A closed form solution may be more efficient, especially for use in optimization and active control problems where the acoustic solution is only one part of the entire mathematical problem. Additionally, a closed form solution may provide more insight into the acoustic behavior.

The Euler–Maclaurin sum formula represents an infinite series as a combination of three parts. These include explicit quantities associated with the limits of the sum, an integral to replace the summation, and a remainder. Applications of the Euler–Maclaurin sum formula have been sparse. Boas and Stutz [3] applied the formula to a few infinite series arising in quantum and statistical mechanics. However, they did not pay much attention to the remainder that can be an important part of the formula. Rawlins [4] applied the formula to an infinite series of logarithmic terms including calculation of the remainder. Riesel [5] showed the basic concepts for applying the formula to a finite double series but no specific examples were calculated.

In this paper, the Euler–Maclaurin sum formula is applied to the infinite series resulting from bounded acoustic waves contained between two parallel plates. A complete analysis and calculation procedure is used to determine the effect of all three parts of the Euler–Maclaurin sum formula. The approach starts with application of the formula to the Green function for the region and concludes with a calculation of the acoustic pressure response for an exponentially decaying transient source (or a polynomial representation of the source). An approximate closed form solution results from applying the formula. Accuracy is demonstrated by comparison with computer calculation of many terms of the infinite series.

2. The Green function

The Green function $g(x, y, z, t)$ is the solution to

$$\nabla^2_{xyz} g(x, y, x, t) - \frac{1}{c^2}\frac{\partial^2 g(x, y, z, t)}{\partial t^2} = -\delta(x - x_0)\delta(y - y_0)\delta(z - z_0)\delta(t), \tag{3}$$

in the region $-\infty < x, y < \infty, 0 \leqslant z, z_0 \leqslant H, 0 \leqslant t < \infty$. We define the vectors $\mathbf{r} = [xy]$, $\mathbf{r}_0 = [x_0 y_0]$ and apply the Laplace transform with respect to time t $(t \to s)$.

$$G(\mathbf{r}, z, s) = \int_0^\infty g(\mathbf{r}, z, t)\mathrm{e}^{-st}\,\mathrm{d}t. \tag{4}$$

Eq. (3) becomes

$$\nabla^2_{\mathbf{r}z} G(\mathbf{r}, z, s) - \left(\frac{s}{c}\right)^2 G(\mathbf{r}, z, s) = -\delta(\mathbf{r} - \mathbf{r}_0)\delta(z - z_0). \tag{5}$$

Defining the distance from source to receiver in the plane of the plates as $r = |\mathbf{r} - \mathbf{r}_0|$, and the following boundary conditions for perfectly reflecting plates:

$$\frac{\partial g}{\partial z} = 0 \quad \text{at } z = 0 \quad \text{and} \quad z = H,$$

the solution to Eq. (5) can be expressed as an infinite sum of the mirror images produced by the plates. Each image at a distance ρ from the source is the free space solution to Eq. (5) given by Hayek [1] and Morse [6]:

$$4\pi G(\rho, s) = \frac{e^{-s\rho/c}}{\rho}.$$

The infinite series of images is given by

$$\begin{aligned} 4\pi G(r, z, s) = {} & \frac{e^{-s/c\sqrt{r^2+(z+z_0)^2}}}{\sqrt{r^2 + (z + z_0)^2}} + \frac{e^{-s/c\sqrt{r^2+(z-z_0)^2}}}{\sqrt{r^2 + (z - z_0)^2}} \\ & + \sum_{k=1}^{\infty} \frac{e^{-s/c\sqrt{r^2+(2kH+z+z_0)^2}}}{\sqrt{r^2 + (2kH + z + z_0)^2}} + \sum_{k=1}^{\infty} \frac{e^{-s/c\sqrt{r^2+(2kH-z-z_0)^2}}}{\sqrt{r^2 + (2kH - z - z_0)^2}} \\ & + \sum_{k=1}^{\infty} \frac{e^{-s/c\sqrt{r^2+(2kH+z-z_0)^2}}}{\sqrt{r^2 + (2kH + z - z_0)^2}} + \sum_{k=1}^{\infty} \frac{e^{-s/c\sqrt{r^2+(2kH-z+z_0)^2}}}{\sqrt{r^2 + (2kH - z + z_0)^2}}. \end{aligned} \tag{6}$$

The form of the Euler–Maclaurin sum formula we use here is give by Apostal [7].

$$\sum_{k=1}^{\infty} f_k = \int_1^{\infty} f(\mu)\,d\mu + \left(\frac{1}{2}\right)[f(1) + f(\infty)] - \int_1^{\infty} \frac{\partial f}{\partial \mu} \sum_{k=1}^{\infty} \frac{\sin(2\pi k\mu)\,d\mu}{\pi k}. \tag{7}$$

Each summation in Eq. (6) has the form

$$S = \sum_{k=1}^{\infty} \frac{e^{-s/cs\sqrt{r^2+(2kH+\varepsilon)^2}}}{\sqrt{r^2 + (2kH + \varepsilon)^2}}, \quad \text{where } \varepsilon = z + z_0,\ z - z_0,\ -z - z_0,\ -z + z_0. \tag{8}$$

Applying the Euler–Maclaurin sum formula, the sum in Eq. (8) is given as

$$\begin{aligned} S = {} & \int_1^{\infty} \frac{e^{-s/c\sqrt{r^2+(2H\mu+\varepsilon)^2}}\,d\mu}{\sqrt{r^2 + (2H\mu + \varepsilon)^2}} + \frac{e^{-s/c\sqrt{r^2+(2H+\varepsilon)^2}}}{2\sqrt{r^2 + (2H + \varepsilon)^2}} \\ & - \sum_{k=1}^{\infty} \frac{1}{\pi k} \int_1^{\infty} \left\{ \frac{\partial}{\partial \mu} \left[\frac{e^{-s/c\sqrt{r^2+(2H\mu+\varepsilon)^2}}}{\sqrt{r^2 + (2H\mu + \varepsilon)^2}} \right] \right\} \sin(2\pi k\mu)\,d\mu, \end{aligned} \tag{9}$$

where $f(\infty) = 0$ by using the domain of convergence $s > 0$ given by Doetsch [8].

In the first integral in Eq. (9), applying the inverse Laplace transform ($s \to t$) gives

$$\int_1^\infty \frac{\delta\left[t - \left(\sqrt{r^2 + (2H\mu + \varepsilon)^2}\right)/c\right] \mathrm{d}\mu}{\sqrt{r^2 + (2H\mu + \varepsilon)^2}}, \tag{10}$$

where δ is the Dirac delta function. (Rules for interchanging the order of the dμ integration and the ds integration from the Laplace inversion are given by Mclachlan [9].)

Letting $\left(\sqrt{r^2 + (2H\mu + \varepsilon)^2}\right)/c = \phi$, the integral in Eq. (10) becomes

$$\frac{1}{2H}\int_{\sqrt{(r^2+(2H+\varepsilon)^2)}/c}^\infty \frac{\delta(t-\phi)\mathrm{d}\phi}{\sqrt{\phi^2 - (r/c)^2}} = \frac{u\left[t - \left(\sqrt{r^2 + (2H + \varepsilon)^2}\right)/c\right]}{2H\sqrt{t^2 - [r/c]^2}}, \tag{11}$$

where the sampling property of the delta function is used (see Ref. [10]) and u is the unit Heaviside step function required to keep the sampling time t within the limits of the integral.

In the second integral in Eq. (9), integration by parts and application of the inverse Laplace transform ($s \to t$) gives

$$2\sum_{k=1}^\infty \int_1^\infty \frac{\delta\left[t - \left(\sqrt{r^2 + (2\pi\mu + \varepsilon)^2}\right)/c\right]\cos(2\pi k\mu)\,\mathrm{d}\mu}{\sqrt{r^2 + (2\pi\mu + \varepsilon)^2}}. \tag{12}$$

Letting $\left(\sqrt{r^2 + (2H\mu + \varepsilon)^2}\right)/c = \phi$, Eq. (12) becomes

$$\frac{1}{H}\sum_{k=1}^\infty \int_{\left(\sqrt{r^2+(2H+\varepsilon)^2}\right)/c}^\infty \frac{\delta(t-\phi)\cos\left\{\frac{\pi k}{H}\left[\sqrt{(\phi c)^2 - r^2} - \varepsilon\right]\right\}\mathrm{d}\phi}{\sqrt{\phi^2 - (r/c)^2}}$$

$$= \frac{u\left[t - \left(\sqrt{r^2 + (2H + \varepsilon)^2}/c\right)\right]}{H\sqrt{t^2 - (r/c)^2}} \sum_{k=1}^\infty \cos\left\{\frac{\pi kc}{H}\left[\sqrt{t^2 - (r/c)^2} - \varepsilon/c\right]\right\}, \tag{13}$$

where the order of summation and integration is reversed.

To investigate the sum in Eq. (13), let $\pi c/H\left[\sqrt{t^2 - (r/c)^2} - \varepsilon/c\right] = a$ and consider (using a result from Knopp [11])

$$\begin{aligned}\sum_{k=1}^\infty \cos(ka) &= \sum_{k=1}^\infty \frac{\mathrm{d}}{\mathrm{d}a}[\sin(ka)] = \pi\frac{\mathrm{d}}{\mathrm{d}a}\left\{\sum_{k=1}^\infty \frac{\sin[2\pi k(a/2\pi)]}{\pi k}\right\} \\ &= -\pi\frac{\mathrm{d}}{\mathrm{d}a}\left\{\left(\frac{\mathrm{a}}{2\pi}\right) - \left[\frac{\mathbf{a}}{\mathbf{2\pi}}\right] - \frac{1}{2}\right\} = -\frac{1}{2} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}a}[\mathbf{a}],\end{aligned} \tag{14}$$

where bold type $[\mathbf{a}/\mathbf{2\pi}]$ and $[\mathbf{a}]$ are the smallest integer parts of $a/2\pi$ and a, respectively.

Now $[\mathbf{a}]$ is the staircase function with an interpretation of average slope equal to one. Thus the result in Eq. (14) approaches zero in an average sense. The second integral in the Euler–Maclaurin sum of Eqs. (7) and (9) may then be considered to be a small remainder given by

$$R_{\varepsilon} = \frac{u\left[t - \left(\sqrt{r^2 + (2H+\varepsilon)^2}\right)/c\right]}{H\sqrt{t^2 - (r/c)^2}}\left(-\frac{1}{2} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}a}[\mathbf{a}]\right). \tag{15}$$

To obtain an approximate representation for the remainder in Eq. (15), consider the upper bound of the remainder from Eq. (7). First use $-\sum_{k=1}^{\infty} \sin(2\pi k\mu)/\pi k = \mu - [\mu] - \frac{1}{2}$ (see Ref. [11]), where $[\mu]$ is the integer part of μ. Then

$$\left|\int_1^{\infty}\left(\mu - [\mu] - \frac{1}{2}\right)\frac{\partial f}{\partial \mu}\,\mathrm{d}\mu\right| \leqslant Max\left|\mu - [\mu] - \frac{1}{2}\right|\left|\int_1^{\infty}\frac{\partial f}{\partial \mu}\,\mathrm{d}\mu\right|$$
$$= \frac{1}{2}\left|\int_1^{\infty}\mathrm{d}f\right| = \frac{1}{2}|f(\infty) - f(1)|. \tag{16}$$

For the function f from Eq. (9), the result in (16) becomes

$$\left|\frac{\mathrm{e}^{-s/c\sqrt{r^2+(2H+\varepsilon)^2}}}{2\sqrt{r^2+(2H+\varepsilon)^2}}\right|. \tag{17}$$

The result in Eq. (17) may be considered as an upper bound on the remainder. Thus the remainder may be viewed as an "order at most" given by

$$R_{\varepsilon} = Laplace^{-1}\left\langle O\left\{\frac{\mathrm{e}^{-s/c\sqrt{r^2+(2H+\varepsilon)^2}}}{2\sqrt{r^2+(2H+\varepsilon)^2}}\right\}\right\rangle \cong \frac{\delta\left(t - \left(\sqrt{r^2+(2H+\varepsilon)^2}\right)/c\right)}{2\sqrt{r^2+(2H+\varepsilon)^2}}. \tag{18}$$

The result in Eq. (18) is the same as the inverse Laplace transform of the second term $f(1)$ in Eqs. (7) and (9).

Using the results of Eqs. (11), (15) and (18), a closed form solution for the Green function in the time domain that comes from the sum in Eq. (9) is given by

$$\begin{aligned} 4\pi g(r,z,t) = {} & \frac{1}{r_1}\delta\left(t - \frac{r_1}{c}\right) + \frac{1}{r_2}\delta\left(t - \frac{r_2}{c}\right) \\ & + \frac{1}{2H\sqrt{t^2-(r/c)^2}}\left\{u\left(t-\frac{r_3}{c}\right) + u\left(t-\frac{r_4}{c}\right) + u\left(t-\frac{r_5}{c}\right) + u\left(t-\frac{r_6}{c}\right)\right\} \\ & + \frac{1}{2r_3}\delta\left(t-\frac{r_3}{c}\right) + \frac{1}{2r_4}\delta\left(t-\frac{r_4}{c}\right) + \frac{1}{2r_5}\delta\left(t-\frac{r_5}{c}\right) + \frac{1}{2r_6}\delta\left(t-\frac{r_6}{c}\right) \\ & + R_{z+z_0} + R_{z-z_0} + R_{-z-z_0} + R_{-z+z_0}, \end{aligned} \tag{19}$$

where $r_1 = \sqrt{r^2+(z+z_0)^2}$, $r_2 = \sqrt{r^2+(z-z_0)^2}$, $r_3 = \sqrt{r^2+(2H+z+z_0)^2}$, $r_4 = \sqrt{r^2+(2H+z-z_0)^2}$, $r_5 = \sqrt{r^2+(2H-z-z_0)^2}$, $r_6 = \sqrt{r^2+(2H-z+z_0)^2}$ and R_{ε} is the remainder given by the exact representation of Eq. (15) or the approximate representation of

Eq. (18). The Green function is composed of discrete delta functions and a group of four dispersion like terms that come from the integral in the Euler–Maclaurin sum formula.

3. Transient response

The transient response for an exponentially decaying source $v(t)$ is considered. To keep the result simple and basic for the purpose of demonstrating the use of the closed form the Green function determined in Section 2, the maximum amplitude of the source is set to one and is located at $z_0 = 0$ with the response calculated at $z = 0$. The source is expressed as

$$v(t) = \mathrm{e}^{-\alpha t}u(t), \quad \text{where } \alpha > 0. \tag{20}$$

For this case, $r_1 = r_2 = r$ and $r_3 = r_4 = r_5 = r_6 = \sqrt{r^2 + (2H)^2} = rh$.

For simplicity, $g = 4\pi g$ is used as the Green function. The transient acoustic pressure response is determined by the convolution

$$p(r,t) = \int_0^t g(t-\tau)v(\tau)\,\mathrm{d}\tau = \int_0^{\mathrm{t}} g(\tau)v(t-\tau)\,\mathrm{d}\tau. \tag{21}$$

The delta functions in the Green function, Eq. (19), easily convolve with $v(t)$ using the second form in Eq. (21):

$$\int_0^t \delta\left(\tau - \frac{r_i}{c}\right)\mathrm{e}^{-\alpha(t-\tau)}\,u(t-\tau)\,\mathrm{d}\tau = \mathrm{e}^{-\alpha\left(t-(r_i)/c\right)}u\left(t - \frac{r_i}{c}\right). \tag{22}$$

To convolve the dispersion like terms in Eq. (19) with $v(t)$, the first form in Eq. (21) is used:

$$\int_0^t \frac{u(t-\tau-r_i/c)\,\mathrm{e}^{-\alpha\tau}}{\sqrt{(t-\tau)^2-(r/c)^2}}\,\mathrm{d}\tau = \int_0^{t-(r_i/c)} \frac{\mathrm{e}^{-\alpha\tau}}{\sqrt{(t-\tau)^2-(r/c)^2}}\,\mathrm{d}\tau\,u\left(t-\frac{r_i}{c}\right). \tag{23}$$

For the specific case, the total acoustic pressure is given by

$$\begin{aligned} p(r,t) = {} & \frac{2\mathrm{e}^{-\alpha(t-(r/c))}}{r}u\left(t-\frac{r}{c}\right) + \frac{2}{H}\int_0^{t-(rh/c)} \frac{\mathrm{e}^{-\alpha\tau}}{\sqrt{(t-\tau)^2-(r/c)^2}}\,\mathrm{d}\tau\,u\left(t-\frac{rh}{c}\right) \\ & + \frac{2\mathrm{e}^{-\alpha(t-(rh/c))}}{rh}u\left(t-\frac{rh}{c}\right) + O\left\{\frac{2\mathrm{e}^{-\alpha(t-(rh/c))}}{rh}u\left(t-\frac{rh}{c}\right)\right\}. \end{aligned} \tag{24}$$

To compare the closed form solution (24) with the infinite series of images, the inverse Laplace transform of Eq. (6) is given by

$$\begin{aligned} p(r,t) = {} & \frac{2\mathrm{e}^{-\alpha(t-(r/c))}}{r}u\left(t-\frac{r}{c}\right) + \frac{Lim}{N\to\infty}4\sum_{k=1}^{N} \frac{\mathrm{e}^{-\left(t-\left(\sqrt{r^2+(2kH)^2}\right)/c\right)}}{\sqrt{r^2+(2kH)^2}} \\ & u\left(t-\frac{\sqrt{r^2+(2kH)^2}}{c}\right). \end{aligned} \tag{25}$$

For a specific numerical case of $H = 10\,\text{ft}$, $r = 30\,\text{ft}$, and $\alpha = 1$, Fig. 1 gives a comparison of the closed form solution of Eq. (24) including remainder and the infinite series solution of Eq. (25). $N = 300$ terms are required for the series solution to converge. The closed form solution with remainder follows the series solution but is slightly higher near the peak area. It can be expected that this may occur. The agreement is very close even at the peak area. Recalling that the exact form of the remainder in Eq. (14) approaches zero in an average sense, the close agreement without remainder seems logical. Otherwise, the approximate form of the remainder used seems to provide an accurate upper bound. Fig. 2 gives a breakdown of the three different terms in the Euler–Maclaurin sum of Eq. (24). The integral part of the sum formula clearly dominates the response with the approximate upper bound remainder being only a small portion of the total response. Fig. 3 repeats the comparison of closed form solution (24) and series solution (25) for the case of $r = 5\,\text{ft}$. The results are consistent with the $r = 30\,\text{ft}$ case.

For the numerical results in Figs. 1–3, the convolution integral in the closed form solution (24) is solved numerically. For the exponential source case, an analytical result may be obtained by

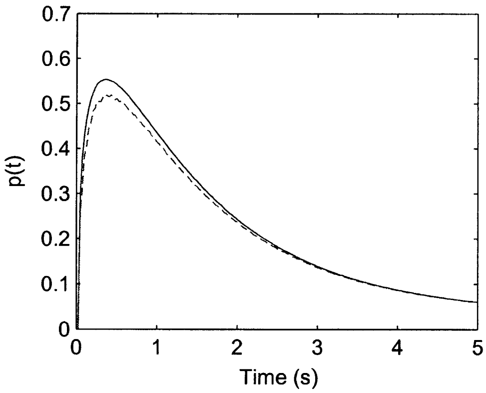

Fig. 1. Euler–Maclaurin sum closed form solution (——) vs. series solution (- - - -) for the case $H = 10\,\text{ft}$ and $r = 30\,\text{ft}$.

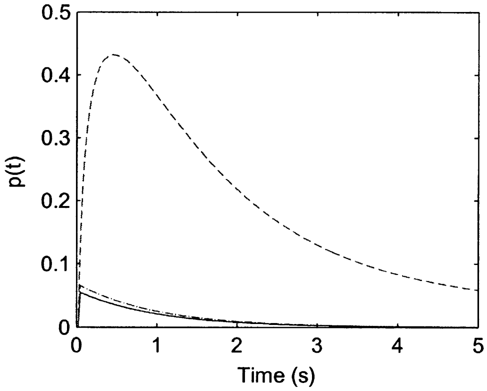

Fig. 2. Three parts of the Euler–Maclaurin sum formula for the case $H = 10\,\text{ft}$ and $r = 30\,\text{ft}$. The r term (_._._.), the integral term (- - - -), and the rh term or remainder (——).

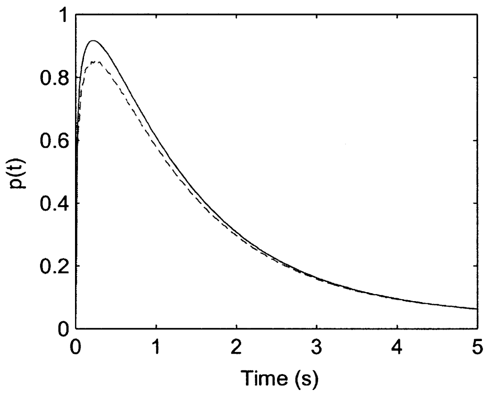

Fig. 3. Euler–Maclaurin sum closed form solution (——) vs. series solution (- - -) for the case $H = 10\,\text{ft}$ and $r = 5\,\text{ft}$.

using a Maclaurin expansion of the exponential. To accomplish this, let $x = t - \tau$ in the integral from Eq. (23). The result is

$$e^{-\alpha t}u\left(t - \frac{r_i}{c}\right)\int_{r_i/c}^{t}\frac{e^{\alpha x}}{\sqrt{x^2 - (r/c)^2}}\,dx. \tag{26}$$

Expanding the exponential in a Maclaurin series

$$e^{\alpha x} = 1 + \alpha x + \frac{\alpha^2 x^2}{2!} + \frac{\alpha^3 x^3}{3!} + \cdots, \tag{27}$$

the first three terms result in exact integrals and the fourth term results in an almost exact solution. Defining $rc = r/c$, these integrals are given by

$$\begin{aligned}
&\int\frac{dx}{\sqrt{x^2 - rc^2}} = \log\left(x + \sqrt{x^2 - rc^2}\right), \quad \int\frac{x\,dx}{\sqrt{x^2 - rc^2}} = \sqrt{x^2 - rc^2},\\
&\int\frac{x^2\,dx}{\sqrt{x^2 - rc^2}} = \frac{x}{2}\sqrt{x^2 - rc^2} - \frac{rc^2}{2}\log\left(x + \sqrt{x^2 - rc^2}\right),\\
&\int\frac{x^3\,dx}{\sqrt{x^2 - rc^2}} = \frac{x^2}{2}\sqrt{x^2 - rc^2} - \frac{rc^2}{2}x\log\left(x + \sqrt{x^2 - rc^2}\right) - \frac{1}{6}\sqrt{(x^2 - rc^2)^3}\\
&\quad + \frac{rc^2}{2}\int\log\left(x + \sqrt{x^2 - rc^2}\right)dx.
\end{aligned} \tag{28}$$

For the specific numerical example and the $r = 30$ case, Fig. 4 shows that the peak for the integral log term in the x^3 term is a factor on the order of 10^4 smaller than the exact parts of the x^3 term and the peak of the entire x^3 term is only about 3% of the peak of the first three terms of the Maclaurin expansion. It also shows that using the first three terms of the Maclaurin expansion for

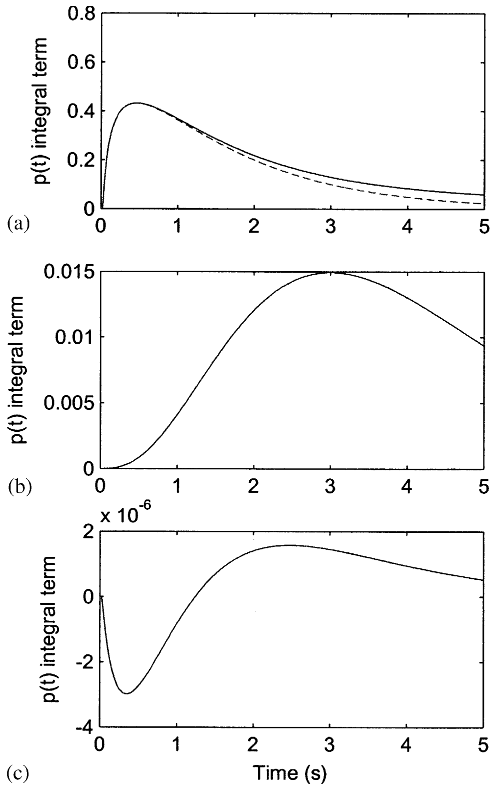

Fig. 4. Euler–Maclaurin sum formula for polynomial expansion of exponential source for the case $H = 10\,\text{ft}$ and $r = 30\,\text{ft}$. (a) Exponential (——) vs. four-term expansion (- - -); (b) x cubed term from Eq. (28); and (c) integral log term from Eq. (28).

the convolution integral (23) give a fairly accurate representation of this integral. Thus for the exponentially decaying source, Eq. (24) reduces to a fairly accurate closed form solution given by

$$\begin{aligned} p(r,t) =& \frac{2e^{-\alpha\left(t-(r/c)\right)}}{r} u\left(t - \frac{r}{c}\right) \\ &+ \frac{2}{H} e^{-\alpha t} \left\{ \begin{array}{c} \left[1 - \left(\frac{\alpha r}{2c}\right)^2\right] \left[\log\left(t + \sqrt{t^2 - \left(\frac{r}{c}\right)^2}\right) - \log\left(\frac{rh}{c} + \frac{2H}{c}\right)\right] \\ +\alpha\sqrt{t^2 - \left(\frac{r}{c}\right)^2}\left(1 + \frac{t\alpha}{4}\right) - \alpha\frac{H}{c}\left(1 + \frac{rh\alpha}{4c}\right) \end{array} \right\} u\left(t - \frac{rh}{c}\right) \\ &+ \frac{2e^{-\alpha\left(t-(rh/c)\right)}}{rh} u\left(t - \frac{rh}{c}\right) + O\left\{\frac{2e^{-\alpha\left(t-(rh/c)\right)}}{rh} u\left(t - \frac{rh}{c}\right)\right\}. \end{aligned} \tag{29}$$

For other types of source time structures, the accuracy of the three-term Maclaurin expansion for the exponential implies that a polynomial representation for a particular function $v(t)$ may lead to similar results. Consider a polynomial representation for $v(t)$ in the region $0 \leqslant t \leqslant t_0$.

$$v(t) = (a_0 + a_1 t + a_2 t^2 + a_3 t^3)[u(t) - u(t - t_0)]. \tag{30}$$

The convolution integral (21) leads to

$$\int_{r_i/c}^{t} \frac{a_0 + a_1(t-x) + a_2(t-x)^2 + a_3(t-x)^3}{\sqrt{x^2 - (r/c)^2}} \mathrm{d}x \; u\left(t - \frac{r_i}{c}\right)$$
$$= \int_{r_i/c}^{t} \frac{(a_0 + a_1 t + a_2 t^2 + a_3 t^3) - (a_1 + 2a_2 t + 3a_3 t^2)x + (a_2 + 3a_3 t)x^2 - (a_3)x^3}{\sqrt{x^2 - (r/c)^2}} \mathrm{d}x \; u\left(t - \frac{r_i}{c}\right). \tag{31}$$

The integral is evaluated in closed form via the results in Eq. (28). One would expect that that the integral term from the x^3 integral would as be small as it is for the exponential source.

4. Remarks

The work presented here indicates that the Euler–Maclaurin sum formula can be successfully used for the infinite series associated with acoustic wave propagation in a bounded region. The inclusion of the remainder analysis with the other two main parts of the formula provides a means of obtaining an accurate closed form solution for the Green function. The application to transient response gives a closed form solution with results very close to the actual series. The small variation from the actual series calculation that converges in 300 terms is acceptable for certain applications. For example, a closed form solution may be more efficient for optimization and active control applications where the acoustic solution is only part of the entire mathematical problem. A closed form solution with the accuracy demonstrated may also provide more insight into the acoustic behavior. It can be expected that the approach may be applied to other physical phenomena governed by the wave equation where finite bounds generate an infinite series.

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